Varianta 068

Subiectul I:

a) $3\sqrt{2}$. b) $5\sqrt{7}$. c) 0. d) $\sqrt{2}$. e) $\frac{\sqrt{6}}{3}$. f) $\cos x=\frac{4}{5}$.

Subiectul II:

1. a) $(f\circ f)(2)=2$. b) $f\ge 1 \Leftrightarrow x^2-6x+9=(x-3)^2\ge 0$ "A". c) P= 3/4. d) 1810. e) x=8.

2. a) $f'(x)=8x-\frac{1}{x^2}$. b) $f'(x)=\frac{8x^3-1}{x^2}$. Pentru $\forall x>1 \Rightarrow x^2>0 \Rightarrow f'(x)>0 \Rightarrow f$ strict crescatoare pe $(1,\infty)$. c) $\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)=7$. d) $f''(x)=8+\frac{2}{x^3}=\frac{8x^3+2}{x^3}, \forall x>0$. Deci f este convexa pe $(0,\infty) \Rightarrow$ nu are nici un punct de inflexiune e) $\int_1^e f(x)dx=\frac{4e^3}{3}-\frac{1}{3}$

Subiectul III:

a) f(0)=$[0]+\left[\frac{1}{n}\right]+\left[\frac{2}{n}\right]+...+\left[\frac{n-1}{n}\right]-[0]=0+0+0+...0-0=0$

b) $f(x+\frac{1}{n})=\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+...+\left[x+\frac{n-1}{n}\right]+[x+1]-[nx+1]=f(x), \forall x\in \mathbf{R}$

c) $0\le x-\frac{k}{n}<\frac{1}{n} \Leftrightarrow \frac{k}{n}\le x<\frac{k+1}{n} \Leftrightarrow k\le nx<k+1$, de unde rezultă $k=[nx]$.

d) Pentru $x\in\left[0,\frac{1}{n}\right)$ avem: $[x]=\left[x+\frac{1}{n}\right]=\left[x+\frac{2}{n}\right]=...=\left[x+\frac{n-1}{n}\right]=[nx]=0$.

e) Conform b) functia f este periodica si $T=\frac{1}{n}$ este perioada a functiei si pentru $x\in\left[0,\frac{1}{n}\right)$ avem $f(x)=0,$ deci $f(x)=0, \forall x\in \mathbf{R}$

f)Punand x=1 in relatia data obtinem s=t.

Presupunem că $a_s\ne b_s$, de exemplu $a_s<b_s$ si alegem $x\in(1-b_s,1-a_s)\subset(0,1)\Rightarrow$ $[x+a_i]=0\Rightarrow\sum_{i=1}^{s}[x+a_i]=0$. Deoarece x, $b_i\in[0,1]\Rightarrow[x+b_i]\ge 0, x+b_i$ fiind pozitiv si $x+b_s>1-b_s+b_s=1$. De aici $\sum_{i=1}^{s}[x+b_i]\ge 1$. Deci $a_s=b_s$ si repetand rationamentul rezulta succesiv $a_{s-1}=b_{s-1},...,a_1=b_1$.

g) $0\le a_1<a_2<....<a_s<1$ şi $[x+a_1]+[x+a_2]+...+[x+a_s]=[sx]]$

Conform c) avem $[x]+[x+\frac{1}{s}]+...+[x+\frac{s-1}{s}]=[sx]$

Rezultă $\sum_{i=1}^{s}[x+a_i]=\sum_{i=1}^{s}[x+\frac{i-1}{s}]$. Deci $\sum_{i=1}^{s}[x+a_i]=\sum_{i=1}^{s}[x+b_i]$ conform pct . f) $\{a_1,a_2,...,a_s\}=\{b_1,b_s,...,b_s\}$.

Subiectul IV:

a) $f_n'(x)=\frac{1}{1!}+\frac{2x}{2!}+\frac{3x^2}{3!}+...+\frac{nx^{n-1}}{n}=f_{n-1}(x),\forall n\in\mathbf{N}$

b) $f_{n+1}(x)=\frac{1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}=f_n(x)+\frac{x^{n+1}}{(n+1)!}=\frac{x^{n+1}}{(n+1)!}+f_n(x),\forall n\in\mathbf{N}^*$

c) $f_2(x)=1+\frac{x}{1!}+\frac{x^2}{2!}=1+x+\frac{x^2}{2},\forall x\in\mathbf{R}$

$f_2(x)\geq\frac{1}{2}\Leftrightarrow 1+x+\frac{x^2}{2}\geq 1\Leftrightarrow x^2+2x+1\geq 0\Leftrightarrow (x+1)^2\geq 0$ adevarat $\forall x\in\mathbf{R}$

d) $x+\int_0^x f_n(t)dt=1+F_n(x)-F_n(0)$, unde F_n este o primitiva a functiei f_n

Conform a) f_{n+1} este o primitiva a lui $f_n\Rightarrow x+\int_0^x f_n(t)dt=1+f_{n+1}(x)-f_{n+1}(0)=f_{n+1}(x)$

e) Pentru n=2, afirmația devine $f_2(x)>0,\forall x\in\mathbf{R}$ adevărat conform c)

$f_2(x)\geq\frac{1}{2}>0,\forall x\in\mathbf{R}$. Demonstram ca daca $f_{2k}(x)>0,\forall x\in R\Rightarrow f_{2k+2}(x)>0,\forall x\in\mathbf{R}$, pentru $k\in\mathbf{N}$

Conform a) $f'_{2k+1}(x)=f_{2k}(x)>0,\forall x\in\mathbf{R}$ => f_{2k+1} strict crescatoare => are cel mult o rădăcină .

Deoarece $\lim\limits_{x\to-\infty}f_{2n-1}(x)=-\infty,\lim\limits_{x\to+\infty}f_{2n+1}(x)=+\infty,\ f_{2k+1}$ continua $\Rightarrow$ are exact o radacina. .

Fie α radacina functiei f_{2k+1}. Avem $f_{2k+2}(x)\geq f_{2k+1}(\alpha)+\frac{\alpha^{2n+2}}{(2n+2)!}>0,\forall x\in\mathbf{R}$.

f) $f'_{2007}(x)=f_{2006}(x)>0,\forall x\in\mathbf{R}\Rightarrow f_{2007}$ strict crescatoare pe $\mathbf{R}$, deci injectiva.

Surjectivitatea rezulta din $\lim\limits_{x\to-\infty}f_{2007}(x)=-\infty,\lim\limits_{x\to+\infty}f_{2007}(x)=+\infty$, si f continua pe $\mathbf{R}$

g) $f_{2008}''(x)=f_{2006}(x)>0,\forall x\in\mathbf{R}\Rightarrow f_{2008}$ convexă pe $\mathbf{R}$.